Solution Sheet 8

1. Please note that the harmonic conjugates are not unique, but they differ by a real constant. This is simply the reflection of the fact that if f is holomorphic so is $f + it$, for any real t.

   - One can always compute the partial derivatives and get:
   $$\frac{\partial^2 u}{\partial x^2} = 6y \ , \ \frac{\partial^2 u}{\partial y^2} = -6y.$$
   Hence it is harmonic everywhere. Another way to do it and simultaneously find a harmonic conjugate is to find a function, such that u is its real part. Consider the function $f(z) = z^3 + \frac{1}{2}z^2$, Let us compute:
   $$f(x+iy) = (x+iy)^3 + \frac{1}{2}(x+iy)^2 = (x^3 - 3xy^2 + \frac{1}{2}x^2 - \frac{1}{2}y^2) + i(3x^2y - y^2 + xy).$$
   Therefore the function we are looking for is if and the harmonic conjugate is $v(x,y) = 3xy^2 - x^3 - \frac{1}{2}x^2 + \frac{1}{2}y^2$.
   - Let us write:
   $$u(x,y) = \sinh(x)\sin(y) = \frac{(e^x - e^{-x})(e^{iy} - e^{-iy})}{4i} = \frac{e^{x+iy} - e^{-x+iy} - e^{x-iy} + e^{-x-iy}}{4i}.$$
   We consider the function $f(z) = \cosh(z)$, then:
   $$f(x+iy) = \frac{e^{x+iy} + e^{-x-iy}}{2}.$$
   Now we take the imaginary part (Recall that $\overline{e^{x+iy}} = e^{x-iy}$):
   $$\mathrm{Im}(f(x+iy)) = \frac{f(x+iy) - \overline{f(x+iy)}}{2i} = \frac{e^{x+iy} + e^{-x-iy} - e^{x-iy} - e^{-x+iy}}{4i}.$$
   Therefore $u = \mathrm{Im}(f(x+iy))$ and hence harmonic in the whole plane. The harmonic conjugate is therefore $-\mathrm{Re}(f(x+iy))$, we shall compute it:
   $$\mathrm{Re}(f(x+iy)) = \frac{f(x+iy) + \overline{f(x+iy)}}{2} = \frac{e^{x+iy} + e^{-x-iy} + e^{x-iy} + e^{-x+iy}}{4} =$$
   $$= \frac{e^x + e^{-x}}{2}\frac{e^{iy} + e^{-iy}}{2} = \cosh(x)\cos(y).$$
   Therefore the harmonic conjugate is $v(x,y) = -\cosh(x)\cos(y)$.
   One can also compute harmonic conjugates with some brute force. We know that $\frac{\partial^1 u}{\partial x^1} = \cosh(x)\sin(y)$ and $\frac{\partial^1 u}{\partial y^1} = \sinh(x)\cos(y)$, Now we need to solve the Cauchy-Riemann equations:
   $$\frac{\partial v}{\partial y} = \cosh(x)\sin(y),$$
   $$\frac{\partial v}{\partial x} = -\sinh(x)\cos(y).$$
   Integrating the first and the second equation, we get that:
   $$-\cosh(x)\cos(y) + g(x) = v(x,y) = -\cosh(x)\cos(y) + h(y).$$
   Here g is a real valued function of only x and similarly h is a real valued function of only y. In particular we get that $g(x) = h(y)$. Applying $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, we get that $g(x) = h(y) = const.$, hence $v(x,y) = -\cosh(x)\cos(y) + c$.

- Let us define $f(z) = e^{z^2}$, hence in Cartesian coordinates:
$$f(x+iy) = e^{(x+iy)^2} = e^{x^2+2ixy-y^2} = e^{x^2-y^2}\left(\cos(2xy) + i\sin(2xy)\right).$$
Therfore u is harmonic and its conjugate is $v(x,y) = e^{x^2-y^2}\sin(2xy)$.

2.
- Since both u and v are harmonic, so is $w = u - v$ and it is 0 on the boundary. By the maximum principle for every $z \in \Omega$:
$$0 = \min_{a\in\partial\Omega} w(a) \leq w(z) \leq \max_{a\partial\Omega} w(a) = 0.$$
Therefore $w(z) = 0$. This implis $u = v$.
- We prove by induction on the order of the derivatives. Since u is harmonic, it is the solution of the homogeneous Laplace equation, namely:
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$
Taking now partial derivative (in x for instance) of this equation, we get:
$$0 = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial x \partial y^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial u}{\partial x}\right).$$
Hence the derivative is also harmonic.
- Let u be harmonic in Ω and let $a \in \Omega$, be such that $u(a) = 0$. For every $r > 0$, such that $\overline{D}_r(a) \subset \Omega$, by the mean value principle, we have:
$$0 = u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{it})dt.$$
Since the function $u(a+re^{it})$ is a continuous function from the circle to $\mathbb{R}$ and the integral is 0 then either it is identically 0 or it has both negative and positive values. In the second case, it has to be 0 at some point on the circle. Since it is true, no matter how small an $r > 0$ we choose, we deduce that a is not an isolated zero.

3. Compute:
$$\frac{az+b}{cz+d} = z \implies az + b = cz^2 + dz \implies cz^2 + (d-a)z - b = 0.$$
It shows, that unless $\varphi(z) = z$, every Möbius transform has at most two fixed points. If $c = 0$, then either $a = d$ and then the only fixed point is ∞ or $a \neq d$ and then there are two fixed points: ∞ and $\frac{b}{d-a}$. If $c \neq 0$, then the fixed points are given by the formula:
$$z_{\pm} = \frac{a - d \pm \sqrt{(d-a)^2 + 4bc}}{2c}.$$
Now the discriminant is $\Delta = d^2 - 2ad + a^2 + 4bc$. Since we can assume that $ad - bc = 1$, we get that $\Delta = (d+a)^2 - 4$. This implies that there is a unique fixed point if $a + d = \pm 2$ and two otherwise.

4.
- Clearly all the specified transforms are Möbius transforms. Now we take a general Möbius transform $\varphi(z) = \frac{az+b}{cz+d}$. If $c = 0$, then $\varphi(z) = \frac{a}{d}z + \frac{b}{d}$, since $a \neq 0$, we get $\varphi(z) = \frac{a}{d}(z + \frac{b}{a}) = d_{\frac{a}{d}}(t_{\frac{b}{a}}(z))$.
Now if $c \neq 0$, we can write $\varphi(z) = \frac{1}{c}\frac{z+b}{z+d/c} = d_{1/c}(\frac{az+b}{z+d/c})$, so we can assume that $\varphi(z) = \frac{az+b}{z+d}$. Now we write:
$$\varphi(z) = \frac{az + b + ad - ad}{z+d} = a + \frac{b - ad}{z+d} = t_a(\frac{b-ad}{z+d}) = t_a(d_{b-ad}(\iota(t_d(z)))).$$

- By the previous part it suffice to show that dillations, translations and the inversion map generalized circles to generalized circles. It is clear for both dilations and translations. So it remains to show that it is true for the inversion. Consider first a circle that doesn't pass through ∞ (i.e. not a line). Hence the equation of the circle is $|z-a|^2 = r^2$. Once we apply the inversion $w = \frac{1}{z}$, we get that the points of the image satisfy the equation $|1-aw|^2 = r^2|w|^2$. Now we write it out explicitly:

  $$0 = |1-aw|^2 - r^2|w|^2 = (1-aw)(1-\overline{aw}) - r^2|w|^2 = 1 - 2\operatorname{Re}(aw) + (|a|^2 - r^2)|w|^2.$$

  Write $w = x+iy$ and $a = u+iv$. Now note that if $|a| = r$, which implies that the circle passes through 0, we get the equation:

  $$1 = 2ux - 2vy.$$

  Therefore the image of the circle is a line. Now assume that $|a| \neq r$, then denote $A = (|a|^2 - r^2)$. We get the equation:

  $$1 - 2ux - 2vy + Ax^2 + Ay^2 = 0.$$

  Now completing squares we get that this is an equation of a circle.
  Now if we have a line $Ax + By = C$, we apply the inversion $\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$. If $C \neq 0$, then 0 does not satisfy the equation and we can write:

  $$A\frac{x}{x^2+y^2} - B\frac{-y}{x^2+y^2} = \frac{C}{x^2+y^2}.$$

  Hence we get that the image satisfies the equation $Au - Bv = C(u^2+v^2)$, this is an equation of the circle. However, if $C = 0$, then the image satisfies $Au - Bv = 0$ and hence it is a line.

5. If φ is of the specified form, it clearly maps $\hat{\mathbb{R}}$ to itself. Now assume φ maps $\hat{\mathbb{R}}$ to itself. Since every Möbius transform is determined by 3 points, let $a, b, c \in \hat{\mathbb{R}}$, be such that $\varphi(a) = 0$, $\varphi(b) = \infty$ and $\varphi(c) = 1$. If neither of the points is ∞, we get that $\varphi(z) = \frac{z-a}{z-b}\frac{c-b}{c-a}$, hence φ is of tthe specified form. If for example $a = \infty$, then $\varphi(z) = \frac{c-b}{z-b}$ and the same conclusion follows.

6. This is basically the Dirichlet problem for the right half-plane. We know how to solve the Dirichlet problem in the unit disc. So we take a Möbius transform that maps the right half-plane to the disc, for example $\varphi(z) = \frac{z-1}{z+1}$ (check it). Then φ^{-1} maps the unit disc o the right half-plane. Define $\tilde{g}(z) = g(\varphi^{-1}(z))$, this is a continuous map on the unit circle. Hence there exists a function $\tilde{u}$ harmonic on the unit unit disc and it coincides with $\tilde{g}$ on the unit cirlce. The function we are looking for is $u(z) = \tilde{u}(\varphi(z))$.

7. Assume φ maps S to itself. Let $a \in \hat{\mathbb{C}}$ be such that $\varphi(a) = 0$, since φ preserves we know that $a \notin S$ and that the point symmetric with respect to S to a is mapped to ∞. If $a = 0$, then the symmetric point is ∞ and we get that $\varphi(z) = cz$, and since φ preserves S, we get that $|c| = 1$, meaning that φ is a rotation. If $a = \infty$ then the symmetric point is 0 and we get that $\varphi(z) = \frac{c}{z}$ and again $|c| = 1$. If $a \in \mathbb{C} \setminus \{0\}$, then the symmetric point is $\frac{1}{\bar{a}}$ and hence $\varphi(z) = c\frac{z-a}{1-\bar{a}z}$. Now if we recall exercise 5 in assignment 1, then we get that if $|a| < 1$, then by the fact that S is preserved, we get again that $|c| = 1$, if $|a| > 1$ then we can always first apply an inversion and get again that $|c| = 1$.

8. - In all of those examples the idea is to map the boundary of D_1 to the boundary of D_2. In this case the boundary is the unit circle (the upper half) and the real axis (a segment on it) and we need to map them to the real and imiginary axes. The transform that does the trick is $\varphi(z) = \frac{1+z}{1-z}$. Note that $\varphi(1) = \infty$, which implies that both the image of the unit circle and the real axis is a line. Now $\varphi(-1) = 0$ and $\varphi(0) = 1$, hence the real axis goes to the real axis (compare 5). Whereas $\varphi(i) = i$ and hence the unit circle is mapped to the imaginary axis. Lastly $\varphi(\frac{i}{2}) = \frac{3+4i}{5} \in D_2$, hence this the transform we need.

- Here we need to map the circle of radius 1 around $1+i$, to the line $\mathrm{Im}(z) = \mathrm{Re}(z)$, such that the interior of the circle will be mapped to the half-plane above the line. We will do it in steps, first map $D_1(1+i)$ to $D_1(0)$ by the map $\varphi_1(z) = z-(1+i)$, then map $D_1(0)$ to the upper half plane by the map $\varphi_2(z) = i\frac{z+1}{1-z}$ (the inverse of $\frac{z-i}{z+i}$) and then map the upper half plane to D_2 by a rotation by $\pi/4$, i.e. $\varphi_3(z) = e^{i\pi/4}z$. Hence the map we want is $\varphi(z) = \varphi_3(\varphi_2(\varphi_1(z)))$(compute its matrix representation).
- Möbius transforms preserve connectedness, so there is no such transform.
- Since those circles intersect at i, we need to map i to the intersection point of the lines, which is clearly ∞. Hence $\varphi(z) = \frac{az+b}{z-i}$. Now by symmetry ∞ has to be mapped to a point on the imaginary axis, and both $\varphi(1+i)$ and $\varphi(-1+i)$ must be symmetric with respect to the line to this point. Assume that $\varphi(\infty) = 0$, then $a = 0$. Hence $\varphi(1+i) = b$ and $\varphi(-1+i) = -b$. Hence if we take $\varphi(z) = \frac{2}{z-i}$ it will do the trick.

9.
   - First we expand this strip to the strip $\{z \in \mathbb{C} \mid -\pi/2 < \mathrm{Im}(z) < \pi/2\}$, via the map $z \mapsto \frac{\pi z}{2}$. Then we apply the exponetial map. So the map we are looking for is $e^{\pi z/2}$.
   - This map is just the identiy.
   - Using the Möbius transform from the previous exercize, we can map this region to the first quadrant. Now apply the map $z \mapsto z^2$ and first quadrant is mapped onto the upper half-plane. Now apply a Möbius transform to map it to the unit disc.
   - Take the principal branch of the logarithm function, $\ln(z) = \ln|z| + i\arg(z)$, where $-\pi < \arg(z) < \pi$. It is conformal on D_1, the image is $\{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0 \ \& \ -\pi < \mathrm{Im}(z) < \pi\}$. Now apply the previous item to map this to the unit disc and map the unit disc to the right half-plane using a Möbius transform.
   - The function $\arcsin(z)$ does the trick. Recall that the function $\arcsin(z)$ is holomorphic on $\mathbb{C} \setminus \{z \in \mathbb{R} \mid |z| \geq 1\}$. It is conformal since its derivative is $\frac{1}{\sqrt{1-z^2}}$. It is straightforward to check that it is the function we need.

10. Note that this strip as we'e seen is conformally equivalent (there exists an invertible conformal map such that its inverse is invertible) to the right half-plane and the right half-plane is equivalent to the unit dic. Let g be the map implementing that conformal equivalence. Define $h(z) = g(f(g^{-1}(z)))$, Note that if $z \in \Omega$ is a fixed point of f, then $w = g(z)$ is a fixed point of h. Hence if f has two fixed points, then so does h. Hence it suffices to prove that this claim is true for a map $h\colon D_1(0) \to D_1(0)$. If a is a fixed point of h, we can use a Möbius transform that maps the disc to itself and a to 0, and assume that $h(0) = 0$. Now we can apply Schwartz' lemma to get that for every $w \in D_1(0)$, $|h(w)| \leq |w|$, now since h has another fixed point, by the Schwartz' lemma again $h(z) = z$.

11. First of all note that $f' = 1 - \frac{1}{z^2}$, hence it is conformal in $\mathbb{C} \setminus \{0, 1, -1\}$. Let $w \in \mathbb{C}$, if $f(z) = w$, then $z^2 - wz + 1 = 0$, hence there are at most two solutions to this equation. If $z \in S_r$, then $z = re^{it}$ and we get:
$$f(z) = re^{it} + \frac{1}{r}e^{-it} = \left(r + \frac{1}{r}\right)\cos(t) + i\left(r - \frac{1}{r}\right)\sin(t).$$
Now write $x = \left(r + \frac{1}{r}\right)\cos(t)$ and $y = \left(r - \frac{1}{r}\right)\sin(t)$. Taking the square we get that:
$$\frac{x^2}{\left(r + \frac{1}{r}\right)^2} + \frac{y^2}{\left(r - \frac{1}{r}\right)^2} = 1.$$
This is an ellipse equation. It is clear now why $S_{1/r}$ is mapped to the same ellipse.